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Good Math

A Geek's Guide to the Beauty of Numbers, Logic, and Computation

The Pragmatic Bookshelf

Dallas, Texas • Raleigh, North Carolina

The Pragmatic Programmers

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This book is dedicated to the memory of my father, Irving Carroll (zt"l). He set me on the road to becoming a math geek, which is why this book exists. More importantly, he showed me, by example, how to be a mensch: by living honestly, with compassion, humor, integrity, and hard work.

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Irrational and Transcendental Numbers

In the history of math, there've been a lot of disappointments for mathematicians. They always start off with the idea that math is a beautiful, elegant, perfect thing. They pursue it, and they eventually discover that it's not.

This leads us to a collection of strange numbers that we need to deal with: the irrational and transcendental numbers. Both were huge disappointments to the mathematicians who discovered them.

What Are Irrational Numbers?

Let's start with the irrational numbers. These are numbers that aren't integers and also aren't a ratio of any two integers. You can't write them as a normal fraction. If you write them as a continued fraction (which we'll describe in 11, *Continued Fractions*, on page ?), then they go on forever. If you write them in decimal form, they go on forever without repeating. They're called irrational because they can't be written as ratios. Many people have claimed that they're irrational because they don't make sense, but that's just a rationalization after the fact.

They do make sense, but they are uncomfortable and ugly to many mathematicians. The existence of irrational numbers means that there are numbers that you cannot write down, and that's an unpleasant fact. You can't ever be precise when you use them: you're always using approximations because you can't write them down exactly. Any time you do a calculation using a representation of an irrational number, you're doing an approximate calculation, and you can only get an approximate answer. Unless you manipulate them symbolically, no calculation that involves them can ever be solved exactly. If you're looking for perfection—for a world in which numbers are precise and perfect—this isn't it.

The transcendental numbers are even worse. *Transcendental numbers* are irrational; but not only can transcendental numbers not be written as a ratio of integers, not only do their decimal forms go on forever without repeating, transcendental numbers are numbers that can't be described by algebraic operations. There are irrational numbers like the square root of 2, which you can easily define in terms of an algebraic equation: it's the value of x in the equation $y = x^2 - 2$ where $y = 0$. You can't write the square root of 2 as a decimal or a fraction, but you can write it with that simple equation. When you're looking at a transcendental number, you can't even do that. There's no finite sequence of multiplications, divisions, additions, subtractions, exponents, and roots that will give you the value of a transcendental number. The square root of 2 is not transcendental, because you can describe it algebraically; but e is.

The *Argh!* Moments of Irrational Numbers

According to legend, the first disappointment involving the irrational numbers happened in Greece around 500 BC. A rather brilliant man by the name of Hippasus, who was part of the school of Pythagoras, was studying roots. He worked out a geometric proof of the fact that the square root of 2 could not be written as a ratio of integers. He showed it to his teacher, Pythagoras. Pythagoras, like so many other mathematicians, was convinced that numbers were clean and perfect and he could not accept the idea of irrational numbers. After analyzing Hippasus's proof and being unable to find any error in it, he became so enraged that he *drowned* poor Hippasus.

A few hundred years later, Eudoxus worked out the basic theory of irrationals, and it was published as a part of Euclid's mathematical texts.

From that point, the study of irrationals pretty much disappeared for nearly two thousand years. It wasn't until the seventeenth century that people really started looking at them again. And once again, it led to disappointment, but at least no one got killed this time.

With the acceptance of irrational numbers, the idea of numbers as something that allowed us to capture the world precisely fell apart. Even something like calculating the circumference of a circle couldn't be done precisely. But mathematicians didn't give up on perfection. They came up with a new idea for what the perfection of numbers in mathematics meant, this time based on algebra. This time they theorized that while you might not be able to write down all numbers as ratios, all numbers must be describable using algebra. Their idea was that for any number, whether integer, rational, or irrational, there was a finite polynomial equation using rational coefficients that had the number as a solution. If they were correct, then any irrational number could be computed by a finite sequence of addition, subtraction, multiplication, division, exponents, and roots.

But it was not to be. The German philosopher, mathematician, and man about town Gottfried Wilhelm Leibniz (1646–1716) was studying algebra and numbers, and he's the one who made the unfortunate discovery that lots of irrational numbers are algebraic but lots of them aren't. He discovered it indirectly by way of the sine function. Sine is one of the basic operations of trigonometry, the ratio of two sides of a right triangle. The sine function is one of the fundamentals of analytic geometry that has real-world implications and is not just a random weird function that someone made up. But Leibniz discovered that you couldn't compute the sine of an angle using algebra. There's no algebraic function that can compute it. Leibniz called sine a transcendental function, since it went beyond algebra. This wasn't quite a transcendental *number*, but it really introduced the idea that there were things in math that couldn't be done with algebra.

Building on the work of Leibniz, the French mathematician Joseph Liouville (1809–1882) worked out that you could easily construct numbers that couldn't be computed using algebra. For example, the constant named after Liouville consists of a string of 0s and 1s where for digit x, 10^{-x} is a 1 if and only if there is some integer n such that $n! = x$.

Once again, mathematicians tried to salvage the beauty of numbers. They came up with a new theory: that transcendental numbers existed, but they needed to be *constructed*. They theorized that while there were numbers that couldn't be computed algebraically, they were all contrived things, things that humans designed specifically to be pathological. They weren't *natural*.

Even that didn't work. Not too much later, it was discovered that e was transcendental. And as we'll see in 6, *e: The Unnatural Natural Number*, on page ?, e is a natural, unavoidable constant. It is absolutely not a contrived creation. Once e was shown to be transcendental, other numbers followed. In one amazing proof, π was shown to be transcendental *using* e. One of the properties that they discovered after recognizing that e was transcendental was that any transcendental number raised to a non-transcendental power was transcendental. Since the value of $e^{i\pi}$ is not transcendental (it's –1), then π must be transcendental.

An even worse disappointment in this area came soon. One of the finest mathematicians of the age, Georg Cantor (1845–1918) was studying the irrationals and came up with the infamous "Cantor's diagonalization," which we'll look at in 16, *Cantor's Diagonalization: Infinity Isn't Just Infinity*, on page ?, which shows that there are more transcendental numbers than there are algebraic ones. Not only are there numbers that aren't beautiful and that can't be used in precise computations, but *most* numbers aren't beautiful and can't be used in precise computations.

What Does It Mean, and Why Does It Matter?

Irrational and transcendental numbers are everywhere. Most numbers aren't rational. Most numbers aren't even algebraic. That's a very strange notion: we can't write most numbers down.

Even stranger, even though we know, per Cantor, that most numbers are transcendental, it's incredibly difficult to prove that any particular number is transcendental. Most of them are, but we can't even figure out which ones!

What does that mean? That our math-fu isn't nearly as strong as we like to believe. Most numbers are beyond us. Here are some interesting numbers that we know are either irrational or transcendental:

- e: transcendental
- π: transcendental
- The square root of 2: irrational, but algebraic
- The square root of x, for all x that are not perfect squares: irrational
- $2^{\text{square root of 2}}$: irrational
- Ω, Chaitin's constant: transcendental

What's interesting is that we really don't know very much about how transcendentals interact; and given the difficulty of proving that something is transcendental, even for the most well-known transcendentals, we don't know much of what happens when you put them together. $\pi+e$; $\pi \times e$; π^e, e^e are all numbers we *don't know* are transcendental. In fact, for $\pi + e$, we don't even know if it's irrational!

That's the thing about these numbers. We have such a weak grasp of them that even things that seem like they should be easy and fundamental, we just do not know how to do. And as we keep studying numbers, it doesn't get any better. For the people who want numbers to make sense, the disappointments keep coming. Not too long ago, an interesting fellow (and former coworker of mine) named Gregory Chaitin (1947–), showed that the irrational numbers are even worse than we thought. Not only are most numbers not rational, not only are most numbers not algebraic, most numbers cannot even be *described in any way*. It's not a big surprise that they can't be written down, because we already know that we can't really write down any irrational number—the best we can do is write a good approximation. In fact, for most numbers, we can't write a description, an equation, or a computer program to generate them. We can't identify them precisely enough to name them. We know they exist, but we're absolutely helpless to describe or identify them in any way at all. It's an amazing idea. If you're interested in it, I highly recommend reading Greg's book, *The Limits of Mathematics [Cha02]*.