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Good Math

A Geek's Guide to the Beauty of Numbers, Logic, and Computation

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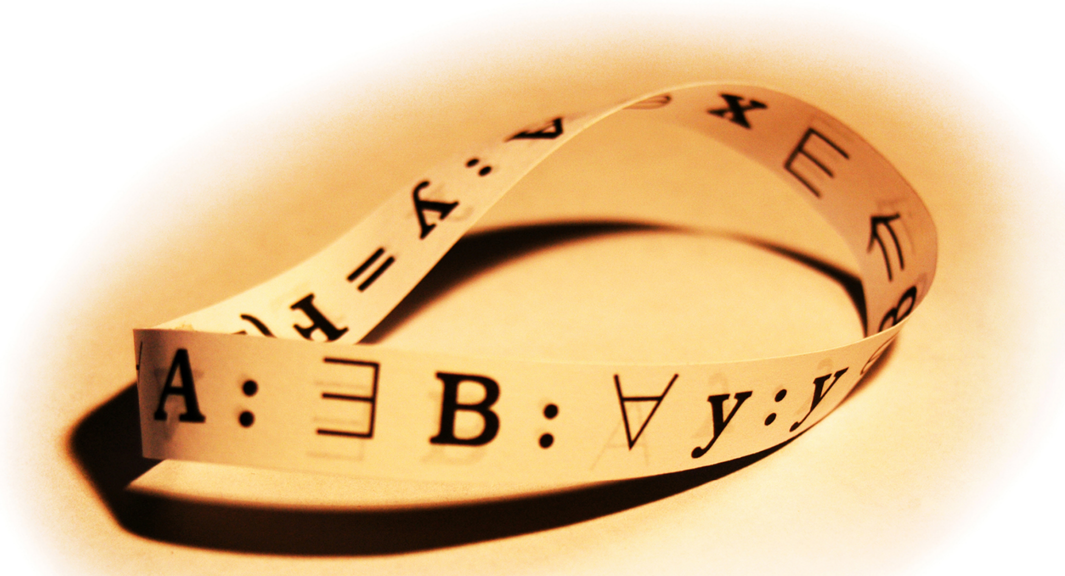

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This book is dedicated to the memory of my father, Irving Carroll (zt"l). He set me on the road to becoming a math geek, which is why this book exists. More importantly, he showed me, by example, how to be a mensch: by living honestly, with compassion, humor, integrity, and hard work.

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Zero

When we look at strange numbers, the starting place has to be zero. Zero may not seem strange to you because you're used to it. But the idea of zero really is strange. Think about what we said numbers mean: if you think in terms of cardinals and ordinals, in terms of counting and position, what does zero mean?

As an ordinal, what does it mean to be the zeroth object in a collection? And what about zero as a cardinal? I can have one something and count it. I can have 10 somethings and count them. But what does it mean to have zero of something? It means that I don't have any of it. So how can I count it?

And yet, without the concept of zero and the numeral 0, most of what we call math would just fall apart.

The History of Zero

In our pursuit of the meaning of zero, let's start with a bit of history. Yes, there's an actual history to zero!

If we were to go back in time and look at when people started working with numbers, we'd find that they had no concept of zero. Numbers really started out as very practical tools, primarily for measuring quantity. They were used to answer questions like "How much grain do we have stored away?" and "If we eat this much now, will we have enough to plant crops next season?" When you think about using numbers in a context like that, a *measurement* of zero doesn't really mean much. A measurement can only make sense if there's something to measure.

Even when math is applied to measurements in modern math, leading zeros in a number—even if they're measured—don't count as significant digits in the measurement. (In scientific measurement, *significant digits* are a way of describing how precise a measurement is and how many digits you can use in computations. If your measurement had two significant digits, then you can't have more than two meaningful digits in the result of any computation based on that measurement.) If I'm measuring some rocks and one weighs 99 grams, then that measurement has only two significant digits. If I use the same scale to weigh a very slightly larger rock and it weighs 101 grams, then my measurement of the second rock has three significant digits. The leading zeros don't count.

We can understand early attitudes about zero by looking back to Aristotle (384–322 BC). Aristotle was an ancient Greek philosopher whose writings are still studied today as the foundations of the European intellectual tradition. Aristotle's thoughts on zero are a perfect example of the reasoning behind why zero wasn't part of most early number systems. He saw zero as a counterpart to infinity. Aristotle believed that both zero and infinity were pure ideas related to the concept of numbers and counting, but that they were not actually numbers themselves.

Aristotle also reasoned that, like infinity, you can't ever get to zero. If numbers are quantities, he thought, then obviously, if you start with one of something and cut it in half, you'll be left with half as much. If you cut it in half again, you'll have one quarter. Aristotle and his contemporaries thought that you could continue that halving process forever: 1/4, 1/8, 1/16, and so on. The amount of stuff you'll have left will get smaller and smaller, closer and closer to zero, but you'll never actually get there.

Aristotle's view of zero does make sense. After all, you can't really have zero of anything, because zero of something is nothing. When you have zero, you don't have a real quantity of stuff. Zero is the absence of stuff.

The first real use of zero wasn't really as a number, but as a digit symbol in numeric notation. The Babylonians had a base-60 number system. They had symbols for numbers from one to 60. For numbers larger than 60, they used a positional system like our decimal numbers. In that positional system, for digit-places with no number, they left a space; that space was their zero. This introduced the idea of a zero as a recordable quantity in some contexts. Later they adopted a placeholder that looked like a pair of slashes (//). It was never used by itself but only as a marking inside multidigit numbers. If the last digit of a number was zero, they didn't write it, because the zero marker was just a placeholder between two non-zero digits to show that there was something in between them. So, for example, the numbers 2 and 120 (in Babylonian base-60, that's 2 × 1 versus 2 × 60) looked exactly the same; you needed to look at the context to see which it was, because they wouldn't write a trailing zero. They had the concept of a notational zero, but only as a separator.

The first real zero was introduced to the world by an Indian mathematician named Brahmagupta (598–668) in the seventh century. Brahmagupta was quite an accomplished mathematician: he didn't just invent zero, but arguably he also invented the idea of negative numbers and algebra! He was the first to use zero as a real number and the first to work out a set of algebraic rules about how zero and positive and negative numbers worked. The formulation he worked out is very interesting; he allowed zero as both a numerator or a denominator in a fraction.

From Brahmagupta, zero spread west (to the Arabs) and east (to the Chinese and Vietnamese). Europeans were just about the last to get it; they were so attached to their wonderful roman numerals that it took quite a while to penetrate: zero didn't make the grade in Europe until about the thirteenth century, when Fibonacci (he of the sequence) translated the works of a Persian mathematician named al-Khwarizmi (from whose name sprung the word *algorithm* for a mathematical procedure). Europeans called the new number system *Arabic* and credited it to the Arabs. As we've seen, the Arabs didn't create Arabic numbers, but it was Arabic scholars, including the famous Persian poet Omar Khayyam (1048–1131), who adopted Brahmagupta's notions *and extended them* to include complex numbers, and it was their writings that introduced these ideas to Europe.

An Annoyingly Difficult Number

Even now, when we recognize zero as a number, it's an annoyingly difficult one. It's neither positive nor negative; it's neither prime nor compound. If you include it in the set of real numbers, then the fundamental mathematical structures like groups that we use to define how numbers apply to things in the world won't work. It's not a unit. Units don't work with it—for any other number, 2 inches and 2 yards mean different things—but that's not true with zero. In algebra, zero breaks a fundamental property called *closure*: without 0, any arithmetic operation on numbers produces a result that is a number. With zero, that's no longer true, because you can't divide by zero. Division is closure for every possible number *except* zero. It's a real obnoxious bugger in a lot of ways. One thing Aristotle was right about: zero is a kind of counterpart to infinity: a concept, not a quantity. But infinity we can generally ignore in our daily lives. Zero we're stuck with.

Zero is a real, inescapable part of our entire concept of numbers. But it's an oddball, the dividing line that breaks a lot of rules. For example, addition and subtraction aren't closed without zero. Integers with addition form a mathematical structure called a *group*—which we'll talk more about in 20, *Group Theory: Finding Symmetries with Sets*, on page ? —that defines what it means for something to be symmetric like a mirror reflection. But if you take away 0, it's no longer a group, and you can no longer define mirror symmetry. Many other concepts in math crumble if we take away zero.

Our notation for numbers is also totally dependent on zero; and it's hugely important for making a polynomial number system work. To get an idea of how valuable it is, just think about multiplication. Without 0, multiplication becomes much, much harder. Just compare long multiplication the way we do it with the way the Romans did multiplication, which I explain in Section 9.3, *Arithmetic Is Easy (But an Abacus Is Easier)*, on page ?.

Because of the strangeness of zero, people make a lot of mistakes involving it.

For example, here's one of my big pet peeves: based on that idea that zero and infinity are relatives, a lot of people believe that one divided by zero is infinity. It isn't. *1/0* doesn't equal anything; the way that we define what division means, it's *undefined*—the written expression *1/0* is a meaningless, invalid expression. You can't divide by 0.

An intuition supporting the fact that you can't divide by zero comes from the Aristotelean notion that zero is a concept, not a quantity. Division is a concept based on quantity, so asking "What is *X* divided by *Y*?" is asking "What quantity of stuff is the right size so that if I take *Y* of it, I'll get *X*?"

If we try to answer that question, we see the problem: what quantity of apples can I take zero of to get one apple? The question makes no sense, and it shouldn't make sense, because dividing by zero makes no sense: *it's meaningless*.

Zero is also at the root of a lot of silly mathematical puzzles and tricks. For example, there's a cute little algebraic pun that can show that 1 = 2, which is based on hiding a division by zero.

Trick: Use Hidden Division by Zero to Show That 1=2.

1. Start with $x = y$.
2. Multiply both sides by x: $x^2 = xy$.
3. Subtract y^2 from both sides: $x^2 - y^2 = xy - y^2$.
4. Factor: $(x + y)(x - y) = y(x - y)$.
5. Divide both sides by the common factor $(x - y)$, giving $x + y = y$.
6. Since $x = y$, we can substitute y for x: $y + y = y$.
7. Simplify: $2y = y$.
8. Divide both sides by y: $2 = 1$.

The problem, of course, is step 5. Because $x - y = 0$, step 5 is equivalent to dividing by zero. Since that's a meaningless thing to do, everything based on getting a meaningful result from that step is wrong—and so we get to "prove" false facts.

Anyway, if you're interested in reading more, the best source of information that I've found is an online article called "The Zero Saga."[1] It covers a bit of history and random chit-chat like this section, but it also provides a detailed presentation of everything you could ever want to know, from the linguistics of the words "zero" and "nothing" to cultural impacts of the concept, to a detailed mathematical explanation of how zero fits into algebras and topologies.

1. http://home.ubalt.edu/ntsbarsh/zero/ZERO.HTM